Discrete Mathematics Quiz 8

Name: ______________________________

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1.1) Select the description that fits the sequence below: 8, 5, 2, 2, 1, -1
a. Non-decreasing but not increasing
b. Non-increasing and decreasing
*c. Non-increasing but not decreasing
d. Non-decreasing and increasing

1.2) What is the common ratio of the following geometric sequence? 27, 9, 3, 1, ...
a. 27
b. 9
c. 3
*d. 1/3

1.3) The sequence $\{f_n\}$ starts with an index of 1 and is defined so that f_n is the largest integer k such that $k^2 \le n$. Which sequence fits the definition of $\{f_n\}$?
a. 1, 4, 9, 16, 25, ...
*b. 1, 1, 1, 2, 2, ...
c. 2, 4, 8, 16, 32, ...
d. 1, 2, 3, 4, 5, ...

2.1) A sequence $\{a_n\}$ is defined as follows: $a_0 = 2$, $a_1 = 1$, and for $n \ge 2$,
$a_n = 3 \cdot a_{n-1} - n \cdot a_{n-2} + 1$. What is a_3?
*a. -2
b. -1
c. 1
d. 2

2.2) A sequence is defined by the recurrence relation $f_n = n \cdot f_{n-1} - f_{n-3}$. How many initial values are required so that the sequence is well defined for all $n \ge 0$?
a. 0

b. 1
c. 2
*d. 3

2.3) A population of mice increases by 10% every year. Define g_n to be the number of mice after n years. Select the recurrence relation that describes the sequence $\{g_n\}$.
a. $g_n = (1.01) \cdot g_{n-1}$
*b. $g_n = (1.1) \cdot g_{n-1}$
c. $g_n = (.01) \cdot g_{n-1} + g_{n-2}$
d. $g_n = (.1) \cdot g_{n-1} + g_{n-2}$

3.1) The inductive step of an inductive proof shows that for $k \geq 0$, if $\sum_{j=0}^{k} 2^j = 2^{k+1} - 1$, then $\sum_{j=0}^{k+1} 2^j = 2^{k+2} - 1$. In which step of the proof is the inductive hypothesis used?

$$
\begin{aligned}
\sum_{j=0}^{k+1} 2^j &= \sum_{j=0}^{k} 2^j + 2^{k+1} && (Step\ 1) \\
&= (2^{k+1} - 1) + 2^{k+1} && (Step\ 2) \\
&= 2 \cdot 2^{k+1} - 1 && (Step\ 3) \\
&= 2^{k+2} - 1 && (Step\ 4)
\end{aligned}
$$

a. Step 1
*b. Step 2
c. Step 3
d. Step 4

3.2) The inductive step of an inductive proof shows that for $k \geq 4$, if $2^k \geq 3k$, then $2^{k+1} \geq 3(k+1)$. In which step of the proof is the inductive hypothesis used?

$$
\begin{aligned}
2^{k+1} &\geq 2 \cdot 2^k && (Step\ 1) \\
&\geq 2 \cdot 3k && (Step\ 2) \\
&\geq 3k + 3k && (Step\ 3) \\
&\geq 3k + 3 && (Step\ 4) \\
&\geq 3(k+1) && (Step\ 5)
\end{aligned}
$$

a. Step 1
*b. Step 2
c. Step 3
d. Step 4

3.3) The inductive step of an inductive proof shows that for $k \geq 4$, if $2^k \geq 3k$, then $2^{k+1} \geq 3(k+1)$. Which step of the proof uses the fact that $k \geq 4 \geq 1$?

$$
\begin{aligned}
2^{k+1} &\geq 2 \cdot 2^k && (Step\ 1)\\
&\geq 2 \cdot 3k && (Step\ 2)\\
&\geq 3k + 3k && (Step\ 3)\\
&\geq 3k + 3 && (Step\ 4)\\
&\geq 3(k+1) && (Step\ 5)
\end{aligned}
$$

a. Step 2
b. Step 3
*c. Step 4
d. Step 5

4.1) Q(n) is a statement parameterized by a positive integer n. The following theorem is proven by induction:
Theorem: For any positive integer n, Q(n) is true.
What must be proven in the inductive step?
a. For any integer $k \geq 1$, Q(k-1) implies Q(k).
b. For any integer $k \geq 1$, Q(k) implies Q(n).
c. For any integer $k \geq 1$, Q(k).
*d. For any integer $k \geq 1$, Q(k) implies Q(k+1).

4.2) The sequence $\{g_n\}$ is defined recursively as follows: $g_0 = 1$, and $g_n = 3 \cdot g_{n-1} + 2n$, for $n \geq 1$. If the theorem below is proven by induction, what must be established in the inductive step?
Theorem: For any non-negative integer n, $g_n = \frac{5}{2} \cdot 3^n - n - \frac{3}{2}$.
a. For $k \geq 0$, if $g_k = 3 \cdot g_{k-1} + 2k$,
then $g_{k+1} = \frac{5}{2} \cdot 3^{k+1} - (k+1) - \frac{3}{2}$.
*b. For $k \geq 0$, if $g_k = \frac{5}{2} \cdot 3^k - k - \frac{3}{2}$,
then $g_{k+1} = \frac{5}{2} \cdot 3^{k+1} - (k+1) - \frac{3}{2}$.
c. For $k \geq 0$, if $g_k = 3 \cdot g_{k-1} + 2k$,
then $g_{k+1} = 3 \cdot g_k + 2(k+1)$.
d. For $k \geq 0$, if $g_k = \frac{5}{2} \cdot 3^k - k - \frac{3}{2}$,
then $g_{k+1} = 3 \cdot g_k + 2(k+1)$.

4.3) Select the mathematical statements to correctly fill in the beginning of the proof of an inductive step below:

We will assume for $k \geq 1$ that 7 evenly divides $6^{2k} - 1$ and will prove that 7 evenly divides $6^{2(k+1)} - 1$. Since, by the inductive hypothesis, 7 evenly divides $6^{2k} - 1$, then 6^{2k} can be expressed as (A?), where m is an integer.

$$
\begin{aligned}
6^{2(k+1)} - 1 &= 6^2 \cdot 6^{2k} - 1 \\
&= (B?) \qquad\qquad \textit{by the ind. hyp.} \\
&= \cdots
\end{aligned}
$$

a. (A): $7m$
 (B): $36(7m) - 1$

*b. (A): $7m + 1$
 (B): $36(7m + 1) - 1$

c. (A): $7m$
 (B): $36(6^{2k}) - 1$

d. (A): $7m + 1$
 (B): $36(6^{2k}) - 1$

5.1) Compute the value of the sum $\sum_{k=-2}^{3} k^2$.

Answer: 19

5.2) Compute the value of the sum $\sum_{k=-2}^{3} (k+1)^2$.

Answer: 31

5.3) Compute the value of the sum $\sum_{k=-2}^{2} (k-1)^2$.

Answer: 15