MA-UY2314 Discrete Mathematics Quiz 9

Name: ____________________________

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1.1) Suppose that in a proof by strong induction, the following statement is the inductive hypothesis:
For $k \geq 6$, P(j) is true for any j in the range {4, 5, ..., k}.
If k is an integer and $k \geq 6$, then which of the following statements can you assume to be true?

a. P(k-1), but not P(k-2) or P(k-3)
b. P(k-3), P(k-2), and P(k-1)
c. Neither P(k-3), P(k-2), nor P(k-1)
*d. P(k-2) and P(k-1), but not P(k-3)

1.2) Let S(n) be a statement parameterized by a positive integer n. Consider a proof that uses strong induction to prove that for all $n \geq 4$, S(n) is true. The base case proves that S(4), S(5), S(6), S(7), and S(8) are all true. Select the correct expressions to complete the statement of what is assumed and proven in the inductive step.

Supposed that for $k \geq (1?)$, S(j) is true for every j in the range 4 through k. Then we will show that (2?) is true.

a. (1): 4
(2): S(k+1)
b. (1): 4
(2): S(j+1)
*c. (1): 8
(2): S(k+1)
d. (1): 8
(2): S(j+1)

1.3) Let S(n) be a statement parameterized by a positive integer n. A proof by strong induction is used to show that for any $n \geq 12$, S(n) is true. The inductive step shows that for any $k \geq 15$, if S(k-3) is true, then S(k+1) is true.
Which fact or set of facts must be proven in the base case of the proof?

a. S(12)
b. S(15)
c. S(12), S(13), and S(14)
*d. S(12), S(13), S(14), and S(15)

2.1) The function SuperPower given below receives two inputs, x and n, and should return $x^{4n-2}$ . x is a real number and n is positive integer.

```
SuperPower(x, n)
        If n = 1, then Return(x^2 )
        y := SuperPower(x, n-1)
Return( ? )
```

What is the correct value for the algorithm to return?
a. $y^4$
b. $x^4$
*c. $y \cdot x^4$
d. $x \cdot y^4$

2.2) The function ComputeSum receives a positive integer $n$ as the input and returns the value

$$ComputeSum(n) = \sum_{j=1}^{n} (j+2)j^2.$$

```
ComputeSum(n)
        If n = 1, then Return(3)
        y := ComputeSum(n-1)
Return( ? )
```

What is the correct value for the algorithm to return?
a. $y$
b. $y + n$
c. $(y+2)y^2$
*d. $y + (n+2)n^2$

2.3) The function ExpPower given below receives two inputs, x and n, and should return $x^{3^n}$. x is a real number and n is a non-negative integer. Note that the exponent of x in the expression $x^{3^n}$ is $3^n$.

```
ExpPower(x, n)
        If n = 0, then Return(?)
        y := ExpPower(x, n-1)
Return( y^3 )
```

What is the correct value for the missing expression?
a. 1
*b. x
c. $x^3$
d. $x^2$

3.1) Which recurrence relation describes a function that has the same asymptotic growth as T(n), defined by the recurrence relation: $T(n) = 2 \cdot T(n/2) + \Theta(n)$

a. $T(n) = T(\lceil n/2 \rceil) + 17$
b. $T(n) = T(\lceil n/2 \rceil) + T([n/2]) + 5^2$
c. $T(n) = T(\lceil n/2 \rceil) + 12n$
*d. $T(n) = T(\lceil n/2 \rceil) + T([n/2]) + (5n + 12)$

3.2) How many times does Loop 1 iterate on input lists A = (2, 7, 11, 16, 21) and B = (3, 4, 6, 9, 13)? Items are removed from the front of the list and added to the back.

```
// Loop 1:
While( size(A) > 0 AND size(B) > 0 )
        If ( front(A) < front(B) )
             x := remove(A)
        Else
             x := remove(B)
        End-if
        add(x, C)
End-while
```

a. 5
b. 7
*c. 8
d. 10

3.3) Which recurrence relation describes the asymptotic complexity of mergesort?

a. $T(n) = T(n/2) + \Theta(1)$
b. $T(n) = 2 \cdot T(n/2) + \Theta(1)$
c. $T(n) = T(n/2) + \Theta(n)$
*d. $T(n) = 2 \cdot T(n/2) + \Theta(n)$

3.4) If mergesort is called with input list (17, 5, 13, 9, 21, 7), then which two lists are sent to the last merge operation?

a. (5, 7, 9) and (13, 17, 21)
b. (17, 5) and (13, 9, 21, 7)
c. (17, 5, 13) and (9, 21, 7)
*d. (5, 13, 17) and (7, 9, 21)

4.1) Select the expression that correctly describes the asymptotic growth of T(n) defined by the recurrence relation:

$$T(n) = 3T(n/2) + \Theta(n^2)$$

*a. $\Theta(n^2)$
b. $\Theta(n^2 logn)$
c. $\Theta(n^{log_2 3})$
d. $\Theta(n^{log_3 2})$

4.2) Select the expression that correctly describes the asymptotic growth of T(n) defined by the recurrence relation:

$$T(n) = 4T(n/2) + \Theta(n^2)$$

a. $\Theta(n^2)$
*b. $\Theta(n^2 logn)$
c. $\Theta(n^3)$
d. $\Theta(n^{log_4 2})$

4.3) Select the expression that correctly describes the asymptotic growth of T(n) defined by the recurrence relation:

$$T(n) = 2T(n/3) + \Theta(1)$$

a. $\Theta(1)$
b. $\Theta(logn)$
c. $\Theta(n)$
*d. $\Theta(n^{log_3 2})$

5.1) Prove that any amount of postage worth 8 cents or more can be made from 3-cent or 5-cent stamps. Hint: by strong induction, check P(8), P(9), P(10), how many base cases do you need?

**Proof.**
By induction on the amount of postage.

**Base case:**

- 8 cents: use a 3-cent and a 5-cent stamp.
- 9 cents: use three 3-cent stamps.
- 10 cents: use two 5-cent stamps.

**Inductive step:** Assume that for k ≥ 10, it is possible to make j cents worth of postage using only 3-cent and 5-cent stamps for any j in the range from 8 through k. Show that it is possible to make k + 1 cents worth of postage using only 3-cent and 5-cent stamps.

Since k ≥ 10, then k - 2 ≥ 8. Therefore k - 2 falls in the range from 8 through k, and by the inductive hypothesis, it is possible to make k - 2 cents worth of stamps using only 3-cent and 5-cent stamps. By adding one 3-cent stamp, the amount of postage becomes (k - 2) + 3 = k + 1. Therefore, it is possible to make k + 1 cents worth of postage using only 3-cent and 5-cent stamps. ■

5.2) Prove that any amount of postage worth 24 cents or more can be made from 7-cent or 5-cent stamps. Hint: by strong induction, check P(24), P(25), P(26), P(27), P(28), how many base cases do you need?

**Proof.**
By induction on the amount of postage.

**Base case:**

- 24 cents: use two 5-cent stamps and two 7-cent stamps.
- 25 cents: use five 5-cent stamps.
- 26 cents: use one 5-cent stamp and three 7-cent stamps.
- 27 cents: use four 5-cent stamps and one 7-cent stamp.

- 28 cents: use four 7-cent stamps.

**Inductive step:** Assume that for $k \geq 28$, it is possible to make j cents worth of postage using only 7-cent and 5-cent stamps for any j in the range from 24 through k. Show that it is possible to make k + 1 cents worth of postage using only 7-cent and 5-cent stamps.

Since $k \geq 28$, then $k - 4 \geq 24$. Therefore k - 4 falls in the range from 24 through k, and by the inductive hypothesis, it is possible to make k - 4 cents worth of stamps using only 7-cent and 5-cent stamps. By adding one 5-cent stamp, the amount of postage becomes $(k - 4) + 5 = k + 1$. Therefore, it is possible to make k + 1 cents worth of postage using only 7-cent and 5-cent stamps. ■

5.3) Prove that any amount of postage worth 4 cents or more can be made from 2-cent or 3-cent stamps. Hint: by strong induction, check P(4), P(5), how many base cases do you need?

**Proof.**

By induction on the amount of postage.

**Base case:**

- 4 cents: use two 2-cent stamps.
- 5 cents: use one 2-cent and one 3-cent stamps.

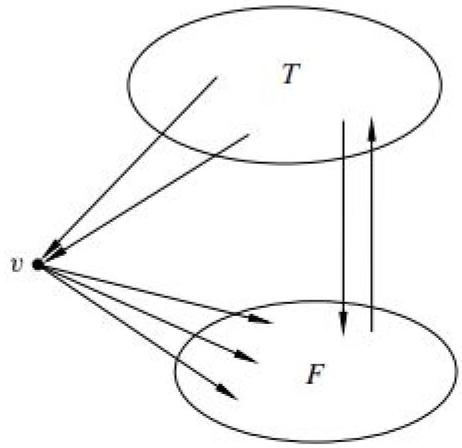
**Inductive step:** Assume that for $k \geq 5$, it is possible to make j cents worth of postage using only 2-cent and 3-cent stamps for any j in the range from 4 through k. Show that it is possible to make k + 1 cents worth of postage using only 2-cent and 3-cent stamps.

Since $k \geq 5$, then $k - 1 \geq 4$. Therefore k - 1 falls in the range from 4 through k, and by the inductive hypothesis, it is possible to make k - 1 cents worth of stamps using only 2-cent and 3-cent stamps. By adding one 2-cent stamp, the amount of postage becomes $(k - 1) + 2 = k + 1$. Therefore, it is possible to make k + 1 cents worth of postage using only 2-cent and 3-cent stamps. ■

5.4) Prove that every round-robin tournament directed graph contains a directed Hamiltonian path by strong induction

Proof. We use strong induction. Let P(n) be the proposition that every tournament graph with n vertices contains a directed Hamiltonian path.

**Base case:** P(1) is trivially true; every graph with a single vertex has a Hamiltonian path consisting of only that vertex.




**Inductive step:** For n  1, we assume that P(1), . . . , P(n) are all true and prove P(n+1). Consider a tournament graph G=(V,E) with n+1 players. Select one vertex v arbitrarily. Every other vertex in the tournament either has an edge to vertex v or an edge from vertex v. Thus, we can partition the remaining vertices into two corresponding sets, T and F , each containing at most n vertices, where $T = \{u \mid u \rightarrow v \in E\}$ and $F = \{u \mid v \rightarrow u \in E\}$ . For example, see the figure above. The vertices in T together with the edges that join them form a smaller tournament. Thus, by strong induction, there is a Hamiltonian path within T . Similarly, there is a Hamiltonian path within the tournament on the vertices in F. Joining the path in T to the vertex v followed by the path in F gives a Hamiltonian path through the whole tournament. As special cases, if T or F is empty, then so is the corresponding portion of the path.

(Source, MIT OpenCourseWare)